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Weak QMV algebras and some ring-like structures

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Abstract In this work, we propose a new quantum structure —weak quantum MV algebras (wQMV algebras)—and define coupled bimonoids and strong coupled bimonoids. We find that the coupled bimonoids and strong coupled bimonoids are ring-like structures corresponding to lattice-ordered wQMV algebras and lattice-ordered QMV algebras, respectively. Using an automated reasoning tool, we give the smallest 4-element wQMV algebra but not a QMV algebra. We also show that lattice-ordered wQMV algebras are the real nondistributive generalization of MV algebras. Certainly, most important properties of quantum MV algebras (QMV algebras) are preserved by wQMV algebras. Furthermore, we can conclude that lattice-ordered wQMV algebras are the simplest unsharp quantum logical structures by far, based on which computation theory could be set up.

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1 Introduction

Quantum logic was introduced in the 1930s of the twentieth century as the logic of quantum mechanics (Birkhoff and Neumann 1936), where projection operators (closed subspaces) of Hilbert space are identified with propositions concerning a quantum mechanical system. Since the set $\mathcal{P}(\mathcal{H})$ of all projection operators of a separable infinite-dimensional Hilbert space is an orthomodular lattice (Kalmbach 1983), orthomodular lattices have been the main model in the study of quantum logic. Any event in $\mathcal{P}(\mathcal{H})$ always satisfies the noncontradiction principle, and such an event is called a sharp event. Quantum logic corresponding to $\mathcal{P}(\mathcal{H})$ is called sharp quantum logic. However, the set of projection operators is not the set of maximal possible events by the statistical rules of quantum theory. In order to meet the need of quantum theory, the set of projection operators is extended to the set $\mathcal{E}(\mathcal{H})$ of positive operators dominated by the identity in Hilbert space (Ludwig 1983). The elements of $\mathcal{E}(\mathcal{H})$ are called effects. Effects correspond to quantum properties that may be disturbed by a certain noise. Since quantum events reflected by $\mathcal{E}(\mathcal{H})$ do not satisfy the noncontradiction principle, they are called unsharp events, and quantum logic corresponding to $\mathcal{E}(\mathcal{H})$ is called unsharp quantum logic (Chiara et al. 2004).

In order to characterize the logical structures of the unsharp proposition systems, many algebraic structures were proposed. Weak orthoalgebras were introduced (Giuntini and Greuling 1989) and further studied as effect algebras (Foulis and Bennett 1994). Another equivalent structure of effect algebras is the D-poset (Kôpka and Chovanec 1994). They are the main algebraic model of unsharp quantum logic.

In the study of unsharp quantum logic, MV algebra is the algebraic counterpart of Łukasiewicz infinite-valued proposition logic Ł$_\infty$ (Chang 1958, 1959). MV algebras play the same role in unsharp quantum logic as Boolean algebras play in sharp quantum logic. Later, another important unsharp quantum structure—quantum MV algebra (QMV algebra)—appeared. QMV algebra was proposed as a non-lattice theoretic generalization of MV algebras and also as a nonidempotent generalization of orthomodular lattices (Giuntini 1996). It is closely related to effect algebras (Dvurečenskig and Pulmannová 2000).

In 1934, Vandiver introduced semirings as an algebraic structure owning two associative binary operations with one distributing over the other. It is well known that semirings are powerful tools in the study of formal languages and automata theory. Certainly, some ring-like algebraic structures are closely connected to structures of quantum logics (Di Nola and Gerla 2005). It was shown that there is a correspondence between coupled semirings and MV algebras (Gerla 2003). A coupled semiring contains an MV algebra, and conversely an MV algebra induces a pair of a lattice-ordered semiring (lc-semiring) and a dual lc-semiring which comprise a coupled semiring. Later, the result was generalized to semirings and pseudo-MV algebras (Shang and Lu 2007). Naturally, we want to know what are the ring-like quantum structures corresponding to QMV algebras?

As an application of those quantum logical structures mentioned above, Ying, Qiu, etc., set up the computation theory based on sharp quantum logic (Qiu 2004, 2007; Ying 2000a, b, 2005) and gave the main difference between classical automata theory and automata theory based on sharp quantum logic. Since unsharp quantum logic is more general than sharp quantum logic, then Shang and Lu studied the computation theory based on unsharp quantum logic (lattice-ordered QMV algebras) (Shang et al. 2009, 2012). And they gave the difference between computation theory based on unsharp quantum logic and computation theory based on sharp quantum logic. Can we find any weaker unsharp quantum structure based on which computation theory could be established?

In this paper, in order to make a step forward to find a weaker quantum logical structure, we generalize QMV algebras to weak QMV algebras (wQMV algebras). It is shown that QMV algebras are a proper subclass of wQMV algebras. Meanwhile, wQMV algebras preserve most important properties such as the monotony, cancelation law and the relation with effect algebras. In Shang et al. (2009), we have proved that if a lattice-ordered QMV algebras satisfy some distributive law, it becomes an MV algebra. However, in this paper, we find that if an MV algebra deletes the same distributive law, it will become a wQMV algebra. From this, we can see wQMV algebras are the real nondistributive generalization of MV algebra. Furthermore, we show that coupled

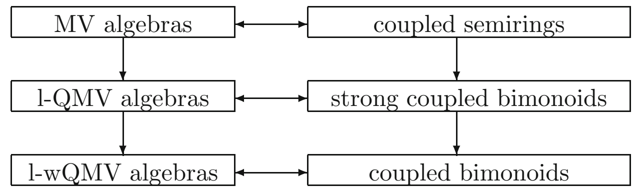

Fig. 1 Circle of relations

bimonoids and strong coupled bimonoids are the ring-like structures corresponding to lattice-ordered wQMV algebras (l-wQMV algebras) and lattice-ordered QMV algebras (l-QMV algebras), respectively, just like coupled semirings are the ring-like structures corresponding to MV algebras. That is, we complete Fig. 1. Similar to the computation theory based on unsharp quantum logic (Shang et al. 2009, 2012), we can successfully set up computation theory based on l-wQMV algebras.

During this research, we found it quite helpful to use automated reasoning tools. In particular, we used the tool SEM (Zhang and Zhang 1995) that can enumerate finite algebras automatically, given the set of axioms characterizing the algebras. With this tool, we were able to generate small examples of QMV algebras and wQMV algebras.

This work is organized as follows. In Sect. 2 we give two equivalent definitions of wQMV algebras and investigate some properties. In Sect. 3, by using the tool of SEM, we generated a 4-element wQMV algebra but not QMV algebra. In Sect. 4, we prove that the center of a wQMV algebra is an MV subalgebra. In Sect. 5 the relation between coupled bimonoids and l-wQMV algebras and the relation between strong coupled bimonoids and l-QMV algebras are established.

2 Weak QMV algebras

As stated in Gudder (1995), a supplement algebra (S-algebra) is a commutative monoid $\mathcal{M} = (M, \oplus, \mathbf{0})$ equipped with a constant element $\mathbf{1}$ and a unary operation $*$ satisfying the following axioms:

(S1) $a \oplus b = b \oplus a$
(S2) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$
(S3) $a \oplus a^* = 1$
(S4) $0 \oplus a = a$
(S5) $1 \oplus a = 1$
(S6) $(a^*)^* = a$

S-algebras are the fundamental structures of quantum MV algebras, effect algebras, and MV algebras, etc.

Definition 2.1 (Giuntini 2005) A quantum MV algebra (QMV algebra) is an S-algebra with (QMV7):

$$a \oplus \left[(a^* \sqcap b) \sqcap (c \sqcap a^*)\right] = (a \oplus b) \sqcap (a \oplus c) \qquad \text{(QMV7)}$$

where $a \odot b := (a^* \oplus b^*)^*$, $a \sqcap b := (a \oplus b^*) \odot b$, $a \sqcup b := (a \odot b^*) \oplus b$.

The characteristic axiom of QMV algebras is (QMV7), which represents a conditional form of distributivity of $\oplus$ over $\sqcap$. In order to introduce a semiring-like structure to QMV algebras, we alter (QMV7) and define a kind of generalized QMV algebra:

Definition 2.2 A weak quantum MV algebra (wQMV algebra) is an S-algebra with (wQMV7):

$$a \oplus \left[(b \sqcap a^*) \sqcap (c \sqcap a^*)\right] = (a \oplus b) \sqcap (a \oplus c) \qquad \text{(wQMV7)}$$

where $\odot$, $\sqcap$ and $\sqcup$ are defined as in Definition 2.1.

As in QMV algebra, we take $\odot$ as more binding than $\oplus$ and define the $\preceq$ relation:

$$a \preceq b \Leftrightarrow a = a \sqcap b$$

In Proposition 2.3 we provide some basic properties that wQMV algebras share with QMV algebras. The verification is straightforward and is omitted here.

Proposition 2.3 *Let $\mathcal{M}$ be a wQMV algebra. The following properties hold:*

(i) $a \odot b = b \odot a$
(ii) $a \odot (b \odot c) = (a \odot b) \odot c$
(iii) $a \odot a^* = \mathbf{0}$
(iv) $a \odot \mathbf{0} = \mathbf{0}$
(v) $a \odot \mathbf{1} = a$
(vi) $a = a \sqcap \mathbf{1} = \mathbf{1} \sqcap a$
(vii) $\mathbf{0} = a \sqcap \mathbf{0} = \mathbf{0} \sqcap a$
(viii) $a = a \sqcap a$
(ix) $(a \sqcup b)^* = a^* \sqcap b^*$
(x) $(a \sqcap b)^* = a^* \sqcup b^*$
(xi) $a \preceq b \Rightarrow a = b \sqcap a$

Proof The same as Theorem 2.3 in Giuntini (1996). □

The monotonicity of several operations also holds in a wQMV algebra:

Proposition 2.4 *Let $\mathcal{M}$ be a wQMV algebra. The following properties hold for any $a, b \in M$:*

(i) $a \oplus b = a \oplus b \sqcap a^*$
(ii) $a \preceq a \oplus b$
(iii) $a \odot b \preceq a$
(iv) $b \preceq a \sqcup b$
(v) $a \sqcap b \preceq b$

Proof (i). There is $a \oplus [(c \sqcap a^*) \sqcap (b \sqcap a^*)] = (a \oplus c) \sqcap (a \oplus b)$ by (wQMV7). Let $c = \mathbf{1}$,

$$\begin{aligned} a \oplus b &= (a \oplus \mathbf{1}) \sqcap (a \oplus b) \\ &= a \oplus (a^* \sqcap (b \sqcap a^*)) \\ &= a \oplus (a^* \oplus (b \sqcap a^*)^*) \odot (b \sqcap a^*) \\ &= a \oplus (a^* \oplus (b^* \sqcup a)) \odot (b \sqcap a^*) \\ &= a \oplus (a^* \oplus (b^* \odot a^*) \oplus a) \odot (b \sqcap a^*) \\ &= a \oplus b \sqcap a^* \end{aligned}$$

(ii). Let $c = \mathbf{0}$ in $a \oplus [(c \sqcap a^*) \sqcap (b \sqcap a^*)] = (a \oplus c) \sqcap (a \oplus b)$, get that $a = a \sqcap (a \oplus b)$ by Proposition 2.3.

(iii). Follows from (i),

$$\begin{aligned} (a \odot b) \sqcap a &= (a \odot b \oplus a^*) \odot a \\ &= (b \sqcup a^*) \odot a \\ &= (b^* \sqcap a \oplus a^*)^* \\ &= (a^* \oplus b^*)^* \\ &= a \odot b \end{aligned}$$

so $a \odot b \preceq a$ by definition.

(iv). $b \preceq a \odot b^* \oplus b = a \sqcup b$ by (ii).

(v). $a \sqcap b = (a \oplus b^*) \odot b \preceq b$ by (iii). □

The monotonicity in QMV algebras could be found in Theorems 2.9 ∼ 2.12 in Giuntini (1996). There is also a verbose definition of QMV algebras; we pick out (QMV8) and (QMV10) in Definition 2.1 of Giuntini (1996) and prove their validity in wQMV algebras.

Theorem 2.5 *Let $\mathcal{M}$ be a wQMV algebra, the following properties hold for any $a, b \in M$:*

(i) $a \sqcup (b \sqcap a) = a$.
(ii) $a \oplus (b \sqcap (a \oplus c)^*) = (a \oplus b) \sqcap (a \oplus (a \oplus c)^*)$.

Proof (i) Following Proposition 2.4 (i) and (ii),

$$\begin{aligned} a \sqcup (b \sqcap a) &= a \odot (b^* \sqcup a^*) \oplus b \sqcap a \\ &= (a^* \oplus b \sqcap a)^* \oplus b \sqcap a \\ &= (a^* \oplus b)^* \oplus (b \oplus a^*) \odot a \\ &= a \sqcup (a \odot b^*) \\ &= (a^* \sqcap (a^* \oplus b))^* \\ &= a \end{aligned}$$

(ii) Replace c with $(a \oplus c)^*$ in (wQMV7). By Proposition 2.4 (i) and (iii),

$$
\begin{aligned}
(a \oplus b) \sqcap (a \oplus (a \oplus c)^*) &= a \oplus ((b \sqcap a^*) \sqcap ((a^* \odot c^*) \sqcap a^*)) \\
&= a \oplus ((b \sqcap a^*) \sqcap (a^* \odot c^*)) \\
&= a \oplus ((b \sqcap a^* \oplus a \oplus c) \odot a^* \odot c^*) \\
&= a \oplus ((b \oplus a \oplus c) \odot a^* \odot c^*) \\
&= a \oplus (b \sqcap (a \oplus c)^*)
\end{aligned}
$$

□

In the following we show that a wQMV algebra is exactly an S-algebra together with (i) and (ii) in Theorem 2.5 as axioms.

Definition 2.6 A WQMV algebra is an S-algebra with

$$a \sqcup (b \sqcap a) = a \tag{WQMV7}$$

$$a \oplus (b \sqcap (a \oplus c)^*) = (a \oplus b) \sqcap (a \oplus (a \oplus c)^*) \tag{WQMV8}$$

As usual we take $\odot$ as more binding than $\oplus$. Define

$$a \preceq b \Leftrightarrow a = a \sqcap b$$

It is straightforward to verify that Proposition 2.3 is valid if $\mathcal{M}$ is a WQMV algebra in the hypothesis.

Proposition 2.7 *Let $\mathcal{M}$ be a WQMV algebra, the following properties hold for any $a, b \in M$:*

(i) $a \oplus (b \sqcap a^*) = a \oplus b$
(ii) $a \odot b \preceq a$.
(iii) *If* $a \preceq b$, *then* $b = b \sqcup a$.
(iv) *If* $a \preceq b$, *then* $b^* \preceq a^*$.
(v) $a = (b^* \sqcap a) \oplus (a \odot b)$.

Proof (i). Let $c = \mathbf{0}$ in (WQMV8), that is $a \oplus (b \sqcap a^*) = a \oplus b$.
(ii). Following (i),

$$
\begin{aligned}
(a \odot b) \sqcap a &= (a \odot b \oplus a^*) \odot a \\
&= (b \sqcup a^*) \odot a \\
&= (b^* \sqcap a \oplus a^*)^* \\
&= (b^* \oplus a^*)^* \\
&= a \odot b
\end{aligned}
$$

so $a \odot b \preceq a$ by definition.
(iii). By (WQMV7), $b = b \sqcup (a \sqcap b) = b \sqcup a$ if $a \preceq b$.
(iv). It follows from (iii) immediately.
(v). $a \odot b \preceq a \Rightarrow a = a \sqcup (a \odot b) = [a \odot (a \odot b)^*] \oplus (a \odot b) = [a \odot (a^* \oplus b^*)] \oplus (a \odot b) = (b^* \sqcap a) \oplus (a \odot b)$. □

Remark 2.8 $a \preceq b \Leftrightarrow b = b \sqcup a$ by Proposition 2.7 (iii) and (iv).

The monotony of the partial relation in QMV algebras is preserved by $\preceq$ in WQMV algebras:

Proposition 2.9 (Monotony) *Let $\mathcal{M}$ be a WQMV algebra. The following properties hold:*

(i) *If $a \preceq b$, then $a \oplus c \preceq b \oplus c$ for any $c \in M$.*
(ii) *If $a \preceq b$, then $a \odot c \preceq b \odot c$ for any $c \in M$.*
(iii) *If $a \preceq b$, then $a \sqcap c \preceq b \sqcap c$ for any $c \in M$.*
(iv) *If $a \preceq b$, then $a \sqcup c \preceq b \sqcup c$ for any $c \in M$.*

Proof (i). By Proposition 2.7 (i) and (v),

$$
\begin{aligned}
(b \oplus c) \sqcup (a \oplus c) &= \left[(b \oplus c) \odot (a^* \odot c^*)\right] \oplus (a \oplus c) \\
&= \left[(a \sqcap b \oplus a^* \odot b \oplus c) \odot (a^* \odot c^*)\right] \\
&\quad \oplus (a \oplus c) \\
&= \left[(a^* \odot b \oplus a \oplus c) \odot (a^* \odot c^*)\right] \\
&\quad \oplus (a \oplus c) \\
&= \left[(a^* \odot b) \sqcap (a^* \odot c^*)\right] \oplus (a \oplus c) \\
&= (a^* \odot b) \oplus a \oplus c \\
&= (b \sqcup a) \oplus c \\
&= b \oplus c
\end{aligned}
$$

so $a \oplus c \preceq b \oplus c$ by Remark 2.8.
(ii). Obtained by (i) and Proposition 2.7 (iv).

(iii) and (iv) Follow from (i) and (ii).

□

Theorem 2.10 *Let $\mathcal{M}$ be a WQMV algebra, then (wQMV7) holds for all $a, b, c \in M$.*

Proof Since $a \preceq (a \oplus b) \sqcap (a \oplus c)$ by Proposition 2.9 (iii), it follows from Proposition 2.7 (i) and (v) that:

$$
\begin{aligned}
(a \oplus b) \sqcap (a \oplus c) &= a \sqcap [(a \oplus b) \sqcap (a \oplus c)] \\
&\quad \oplus [(a \oplus b) \sqcap (a \oplus c)] \odot a^* \\
&= a \oplus \big(a \oplus b \oplus (a \oplus c)^*\big) \\
&\quad \odot (a \oplus c) \odot a^* \\
&= a \oplus \big(b \sqcap a^* \oplus a \oplus (a^* \odot c^*)\big) \\
&\quad \odot (a \oplus c) \odot a^* \\
&= a \oplus \big(b \sqcap a^* \oplus c^* \sqcup a\big) \odot (c \sqcap a^*) \\
&= a \oplus \left[(b \sqcap a^*) \sqcap (c \sqcap a^*)\right]
\end{aligned}
$$

□

Theorems 2.5 and 2.10 indicate that Definition 2.1 coincides with Definition 2.6; wQMV algebras are exactly WQMV algebras.

Since in wQMV algebras as well as in QMV algebras the operation $\sqcap$ is not associative, two kinds of weak associativity are presented based on wQMV algebras:

Proposition 2.11 *Let $\mathcal{M}$ be a wQMV algebra, the following properties hold for any $a, b, c \in M$:*

(i) $(a \oplus b) \sqcup (a \oplus c) = a \oplus [(b \sqcap a^*) \sqcup c]$.
(ii) $(a \sqcap b) \sqcap c = (a \sqcap b) \sqcap (b \sqcap c)$.
(iii) $a \sqcap (b \sqcap c) = (a \sqcap c) \sqcap (b \sqcap c)$.

Proof (i).

$$\begin{aligned}(a \oplus b) \sqcup (a \oplus c) &= a \sqcap [(a \oplus b) \sqcup (a \oplus c)] \\ &\quad \oplus [(a \oplus b) \sqcup (a \oplus c)] \odot a^* \\ &= a \oplus \big[(a \oplus b) \odot (a \oplus c)^* \oplus a \oplus c\big] \odot a^* \\ &= a \oplus \big[(a \oplus b) \odot a^* \odot c^* \oplus a \oplus c\big] \odot a^* \\ &= a \oplus \big[(b \sqcap a^* \oplus a) \odot a^* \odot c^* \oplus a \oplus c\big] \odot a^* \\ &= a \oplus \big[(b \sqcap a^*) \sqcap a^* \odot c^* \oplus a \oplus c\big] \odot a^* \\ &= a \oplus \big[(b \sqcap a^*) \odot c^* \oplus c \oplus a\big] \odot a^* \\ &= a \oplus \big[(b \sqcap a^*) \sqcup c \oplus a\big] \odot a^* \\ &= a \oplus \big[(b \sqcap a^*) \sqcup c\big] \sqcap a^* \\ &= a \oplus \big[(b \sqcap a^*) \sqcup c\big]\end{aligned}$$

(ii).

$$\begin{aligned}(a \sqcap b) \sqcap (b \sqcap c) &= \big[(a \sqcap b) \oplus (b^* \sqcup c^*)\big] \odot (b \sqcap c) \\ &= \big[(a \sqcap b) \oplus (b^* \odot c) \oplus c^*\big] \odot (b \oplus c^*) \odot c \\ &= \big[((a \sqcap b) \oplus c^*) \sqcap (b \oplus c^*)\big] \odot c \\ &= ((a \sqcap b) \oplus c^*) \odot c \\ &= (a \sqcap b) \sqcap c\end{aligned}$$

(iii).

$$\begin{aligned}a \sqcap (b \sqcap c) &= (a \oplus (b \sqcap c)^*) \odot (b \sqcap c) \\ &= (a \oplus b^* \odot c \oplus c^*) \odot (b \oplus c^*) \odot c \\ &= \big[(a \oplus c^*) \sqcap (b \oplus c^*)\big] \odot c \\ &= \big[(c^* \oplus a) \sqcap (c^* \oplus b)\big] \odot c \\ &= \big[c^* \oplus ((a \sqcap c) \sqcap (b \sqcap c))\big] \odot c \\ &= [(a \sqcap c) \sqcap (b \sqcap c)] \sqcap c \\ &= (a \sqcap c) \sqcap (b \sqcap c)\end{aligned}$$

□

Theorem 2.12 *In a wQMV algebra $\preceq$ is a partial-order relation.*

Proof Reflexivity: $a \sqcap a = a \Rightarrow a \preceq a$.

Antisymmetry: If $a \preceq b$, namely $b^* \preceq a^*$, then $b^* \odot a \preceq a^* \odot a = \mathbf{0}$. So $a = b^* \odot a \oplus b \sqcap a = b \sqcap a = b$.

Transitivity: Suppose $a \preceq b$ and $b \preceq c$. Then $a \sqcap c = (a \sqcap b) \sqcap c = (a \sqcap b) \sqcap (b \sqcap c) = a \sqcap b = a$ by Proposition 2.11 (ii). □

Theorem 2.13 (**Cancelation law**) *Let $\mathcal{M}$ be a wQMV algebra. For any $a, b, c \in M$ such that $a \preceq c^*$ and $b \preceq c^*$,*

(i) *If $a \oplus c \preceq b \oplus c$, then $a \preceq b$.*
(ii) *If $a \oplus c = b \oplus c$, then $a = b$.*

Proof (i). Since $a \preceq c^*$ and $b \preceq c^*$, $a = a \sqcap c^* = (a \oplus c) \odot c^* \preceq (b \oplus c) \odot c^* = b \sqcap c^* = b$.
(ii). It follows from (i).

□

Theorem 2.14 *Let $\mathcal{M}$ be a wQMV algebra, then $\mathcal{M}$ is a QMV algebra if and only if $a \oplus b = a \oplus (a^* \sqcap b)$ for any $a, b \in M$.*

Proof "If part." Only need to prove (QMV7),

$$\begin{aligned}(a \oplus b) \sqcap (a \oplus c) &= a \sqcap [(a \oplus b) \sqcap (a \oplus c)] \oplus [(a \oplus b) \sqcap (a \oplus c)] \odot a^* \\ &= a \oplus (a \oplus b \oplus (a \oplus c)^*) \odot (a \oplus c) \odot a^* \\ &= a \oplus \big(a^* \sqcap b \oplus a \oplus (a^* \odot c^*)\big) \odot (a \oplus c) \odot a^* \\ &= a \oplus (a^* \sqcap b \oplus c^* \sqcup a) \odot (c \sqcap a^*) \\ &= a \oplus \big[(a^* \sqcap b) \sqcap (c \sqcap a^*)\big]\end{aligned}$$

"Only if part." See Lemma 2.2 of Giuntini (2005). □

The equation presented in the theorem above is a kind of conditional distributive law of $\oplus$ over $\sqcap$.

For wQMV algebras, a generalization of Theorem 2.14 in Giuntini (1996) is obtained:

Theorem 2.15 *Let $\mathcal{M}$ be a wQMV algebra, the following conditions are equivalent:*

(i) *$\mathcal{M}$ is an MV algebra.*
(ii) *$\forall a, b \in M$: If $a^* \oplus b = \mathbf{1}$, then $a \preceq b$.*

Proof (i) implies (ii). See Theorem 2.14 of Giuntini (1996).
(ii) implies (i). If $a^* \oplus b = \mathbf{1}$, then $b \sqcap a = (a^* \oplus b) \odot a = a$. Conversely, if $b \sqcap a = a$, then $a^* \oplus (b \sqcap a) = a^* \oplus b = \mathbf{1}$. Hence $a^* \oplus b = \mathbf{1}$ is equivalent to $b \sqcap a = a$.

For any $a, b \in M$, $a \sqcap b = a^* \odot (a \sqcap b) \oplus a \sqcap (a \sqcap b) = (a \oplus b^*) \odot b \odot a^* \oplus a \sqcap (a \sqcap b) = a \sqcap (a \sqcap b)$, which implies $a \sqcap b \preceq a$ by the hypothesis. So $a \sqcap b = (a \sqcap b) \sqcap a = (a \sqcap b) \sqcap (b \sqcap a)$ by Proposition 2.11 (ii). It follows that $a \sqcap b \preceq b \sqcap a$. In the same way, $b \sqcap a \preceq a \sqcap b$, so $a \sqcap b = b \sqcap a$. It implies $a \oplus b = a \oplus b \sqcap a^* = a \oplus a^* \sqcap b$, so $\mathcal{M}$ is a QMV algebra by Theorem 2.14, and also an MV algebra. □

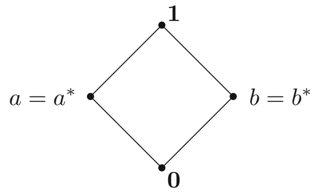

Fig. 2 $\mathfrak{M}_4$

Let $\mathcal{M}$ be a wQMV algebra. We say $\mathcal{M}$ is a lattice-ordered wQMV algebra (l-wQMV algebra for short) if $\mathcal{M}$ forms a lattice with the partial-order $\preceq$. When $\mathcal{M}$ is lattice-ordered, the distributivity of $\oplus$ over $\wedge$ makes $\mathcal{M}$ be an MV algebra.

Theorem 2.16 *Let $\mathcal{M}$ be a l-wQMV algebra, the following conditions are equivalent:*

(i) *$\mathcal{M}$ is an MV algebra.*
(ii) *For all $x, y, z \in M$, $(x \oplus y) \wedge (x \oplus z) = x \oplus (y \wedge z)$.*

Proof (i) implies (ii). Refer to Dvurečenskig and Pulmanová (2000).
(ii) implies (i). For any $a, b \in M$, assume $a' \oplus b = \mathbf{1}$. Let $x = a'$, $y = b$, $z = a$, then $(a' \oplus b) \wedge (a' \oplus a) = a' \oplus (b \wedge a)$, namely $a' \oplus b = a' \oplus (b \wedge a)$. By $a' \oplus b = 1$, then $a' \oplus (b \wedge a) = a' \oplus a$. Since $b \wedge a \preceq a$, $a \preceq a$, we have $b \wedge a = a$ by the cancellation law. That is, $a \preceq b$. So $\mathcal{M}$ is an MV algebra by Theorem 2.15. □

Axiom (wQMV7) is a kind of weak distributivity law. In fact $a \boxplus [(b \sqcap a^*) \sqcap (c \sqcap a^*)] = (a \boxplus b) \sqcap (a \boxplus c) = (a \boxplus (b \sqcap a^*)) \sqcap (a \boxplus (c \sqcap a^*))$. Generally, the distributivity law does not hold in a wQMV algebra. However, wQMV algebras endowed with some kind of "distributivity law" turn out to be QMV algebras:

Theorem 2.17 *Let $\mathcal{M}$ be a wQMV algebra. It is a QMV algebra if one of the following conditions is satisfied.*

(i) *$(a \oplus b) \sqcap (a \oplus c) = a \oplus (b \sqcap c)$ for all $a, b, c \in M$.*
(ii) *$(a \oplus b) \sqcap (a \oplus c) = a \oplus (c \sqcap b)$ for all $a, b, c \in M$.*

Proof (i). Take $b = a^*$, then $a \oplus (a^* \sqcap c) = a \oplus c$ for any $a, c \in M$. Then $\mathcal{M}$ is a QMV algebra by Theorem 2.14.
(ii). Take $c = a^*$ and follow the same proof of (i). □

Example 2.18 Consider the smallest QMV algebra $\mathfrak{M}_4$ in Fig. 2 which is not an MV algebra. The operation is defined as $a \oplus b = \mathbf{1}$.

It is easy to check that (i) and (ii) in Theorem 2.17 are satisfied. Since QMV algebras are special wQMV algebras, it follows that there exists a wQMV algebra satisfying (i), (ii) in Theorem 2.17 but not an MV algebra.

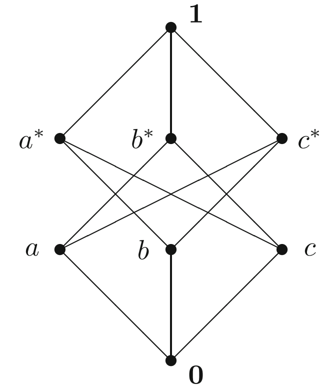

Fig. 3 A wQMV algebra but not a QMV algebra

Theorem 2.19 *Let $\mathcal{M}$ be a wQMV algebra. If $a \preceq a \sqcup b$ or $a \sqcap b \preceq a$ for any $a, b \in M$, then $\mathcal{M}$ is an MV algebra.*

Proof The condition "$a \preceq a \sqcup b$ for all a, b" is equivalent to "$a \sqcap b \preceq a$ for all a, b." Since $(a \sqcap b) \sqcap (b \sqcap a) = (a \sqcap b) \sqcap a = a \sqcap b$, so $a \sqcap b \preceq b \sqcap a$ $\forall a, b \in M$. It follows that $a \sqcap b = b \sqcap a$, $\forall a, b$, and $\mathcal{M}$ is an MV algebra. □

In the following, we give an example which is a wQMV algebra but not a QMV algebra.

Example 2.20 Consider $\mathcal{M} = (\{a, b, c, a^*, b^*, c^*, \mathbf{0}, \mathbf{1}\}, \oplus, ^*, \mathbf{0}, \mathbf{1})$ in Fig. 3.

The operation $\oplus$ is defined as follows:

$\oplus$	a	b	c	a^*	b^*	c^*
a	a	c^*	b^*	$\mathbf{1}$	$\mathbf{1}$	c^*
b	c^*	b	a^*	a^*	$\mathbf{1}$	c^*
c	b^*	a^*	c	a^*	b^*	$\mathbf{1}$
a^*	$\mathbf{1}$	a^*	a^*	a^*	$\mathbf{1}$	$\mathbf{1}$
b^*	$\mathbf{1}$	$\mathbf{1}$	b^*	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
c^*	c^*	c^*	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	c^*

It is straightforward to verify that $\mathcal{M}$ is a wQMV algebra.

$$(a \oplus b^*) \sqcap (a \oplus b) = \mathbf{1} \sqcap c^* = c^*$$
$$a \oplus \left[(b^* \sqcap a^*) \sqcap (b \sqcap a^*)\right] = a \oplus (a^* \sqcap b) = c^*$$
$$a \oplus \left[(a^* \sqcap b^*) \sqcap (b \sqcap a^*)\right] = a \oplus (c \sqcap b) = a$$

Thus $\mathcal{M}$ is not a QMV algebra. Reexamine the condition in Theorem 2.14, $a \oplus a^* \sqcap b^* = a \oplus (a^* \oplus b) \odot b^* = a \oplus a^* \odot b^* = a \oplus c = b^* \neq a \oplus b^* = \mathbf{1}$.

Example 2.21 Let $\mathcal{E}(\mathcal{H})$ be the class of all effects in a Hilbert space $\mathcal{H}$, that is the set of all bounded linear operators between $\mathbf{0}$ and $\mathbf{1}$, where $\mathbf{0}$ is the null operator; $\mathbf{1}$ is the identity operator. Define $\oplus$ and $*$ for $E, F \in \mathcal{E}(\mathcal{H})$ as:

$$E \oplus F = \begin{cases} E + F, & \text{if } E + F \in \mathcal{E}(\mathcal{H}) \\ \mathbf{1}, & \text{otherwise} \end{cases}$$
$$E^* = \mathbf{1} - E$$

where $+$ and $-$ are the usual sum and difference operations in $\mathcal{E}(\mathcal{H})$. The structure $(\mathcal{E}(\mathcal{H}), \oplus, *, \mathbf{0}, \mathbf{1})$ is a wQMV algebra. The relation $\preceq$ defined in a wQMV algebra coincides with the usual partial order in $\mathcal{E}(\mathcal{H})$.

It is well known that the set of all idempotent elements in a QMV algebra is an orthomodular lattice, but this is not true for wQMV algebras. Define $M_I = \{a \in M | a = a \oplus a\}$ to be the set of all idempotent elements of wQMV algebra $\mathcal{M}$. In fact M_I is not closed under *. For instance, in Example 2.20 b is idempotent but b^* is not. It is easy to see that M_I is an upper semilattice where $a \oplus b$ is the least upper bound of a and b. For the idempotent elements in a wQMV algebra, we have the following conclusion:

Proposition 2.22 *Let $\mathcal{M}$ be a wQMV algebra. For any $a \in \mathcal{M}$, $a \sqcap a^* = \mathbf{0}$ iff $a \oplus a = a$.*

Proof If $a \sqcap a^* = \mathbf{0}$, then $a = a \sqcap a^* \oplus a = (a \oplus a) \odot a^* \oplus a = (a \oplus a) \sqcup a = a \oplus a$ by Propositions 2.4 and 2.7. Conversely, if $a \oplus a = a$, then $a \sqcap a^* = (a \oplus a) \odot a^* = a \oplus a^* = \mathbf{0}$. □

Definition 2.23 A quasi-linear wQMV algebra is a wQMV algebra that satisfies the following condition:

$$a \sqcap b = \begin{cases} a, & \text{if } a \preceq b, \\ b, & \text{otherwise.} \end{cases}$$

The quasi-linearity of wQMV algebras is equivalent to some kind of cancelation law.

Proposition 2.24 *Let $\mathcal{M}$ be a wQMV algebra. The following conditions are equivalent:*

(i) *$\mathcal{M}$ is quasi-linear.*
(ii) *$\forall a, b, c \in \mathcal{M}$, $a \oplus c = b \oplus c \neq \mathbf{1}$ implies $a = b$.*

Proof (i) implies (ii): By Proposition 2.4, $a \oplus c = a \oplus (c \sqcap a^*)$. Since $\mathcal{M}$ is quasi-linear, $c \sqcap a^* = c$ or a^*, then $a \oplus c \neq \mathbf{1}$ ensures $c \sqcap a^* = c$, that is $c \preceq a^*$. Similarly, $c \preceq b^*$. So $a = b$ by cancelation law.
(ii) implies (i): $\forall a, b \in \mathcal{M}$, $a^* \oplus b = a^* \oplus (b \sqcap a)$. If $a^* \oplus b \neq \mathbf{1}$, then $b \sqcap a = b$. If $a^* \oplus b = \mathbf{1}$, then $b \sqcap a = (b \oplus a^*) \odot a = a$.

□

Actually quasi-linearity turns a wQMV algebra into a QMV algebra.

Proposition 2.25 *Quasi-linear wQMV algebras and quasi-linear QMV algebras coincide.*

Proof Clearly all quasi-linear QMV algebras are quasi-linear wQMV algebras. On the other hand, if $a^* \preceq b$, then $a \oplus (a^* \sqcap b) = a \oplus a^* = \mathbf{1} = a \oplus b$. Otherwise $a \oplus (a^* \sqcap b) = a \oplus b$. So any quasi-linear wQMV algebra is a quasi-linear QMV algebra by Theorem 2.14. □

Therefore an effect algebra could be converted to a quasi-linear wQMV algebra and vice versa (Chiara et al. 2004).

3 Automatic generation of finite algebras

As mentioned in Sect. 1, we used a tool called SEM (Zhang and Zhang 1995) to generate small algebras automatically. The input of the tool consists of a set of axioms specifying the algebraic structure and the size of the structure.

As an instance, let us see how to find a QMV algebra using the tool. The following is the input file:

```
4.

f(x,y) = f(y,x).
f(x,f(y,z)) = f(f(x,y),z).
f(x,i(x)) = 1.
f(x,0) = x.
i(i(x)) = x.
f(x,1) = 1.

g(x,y) = i(f(i(x),i(y))).
m(x,y) = g(f(x,i(y)),y).
j(x,y) = f(g(x,i(y)),y).

 f(x,m(m(i(x),y),m(z,i(x))))
= m(f(x,y),f(x,z)).
```

Here `f` corresponds to the binary operation $\oplus$, `g` corresponds to the binary operation $\odot$, and `i` corresponds to the unary operation $*$; `m` is the meet operation ($\sqcap$), and `j` is the join operation ($\sqcup$). The last line in the above input corresponds to Eq. (QMV7). The first line specifies the size of the algebra. That is, it has 4 elements.

Given the above input, the tool SEM can generate several 4-element QMV algebras automatically.

If we would like to construct wQMV algebras, we just replace the last line of the input file by the following:

```
f(x,m(m(y,i(x)),m(z,i(x))))
   = m(f(x,y),f(x,z)).
```

It corresponds to Eq. (wQMV7).

By comparing the set of 4-element QMV algebras and the set of 4-element wQMV algebras, we found that the fol-

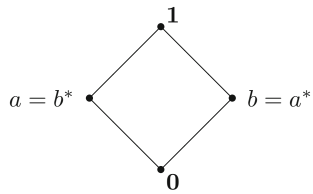

Fig. 4 A smallest wQMV algebra but not a QMV algebra

lowing algebra is a wQMV algebra but not a QMV algebra (Fig. 4).

In addition to the elements 0 and 1, it has two elements: a and b, with $a^* = b$ and $b^* = a$. The operation $\oplus$ is defined as follows:

$\oplus$	0	1	a	b
0	0	1	a	b
1	1	1	1	1
a	a	1	1	1
b	b	1	1	b

This "counterexample" is much smaller than what we give in Example 2.20.

4 Center of wQMV algebras

As we know that a QMV algebra is an MV algebra if and only if the operation $\sqcap$ is commutative. In general, the center of a QMV algebra is an MV subalgebra and therefore a distributive De Morgan lattice (Giuntini 1998). In this section, we define commutativity in wQMV algebras and prove that the center of a wQMV algebra is also an MV subalgebra and a distributive De Morgan lattice.

Definition 4.1 Let $\mathcal{M}$ be a wQMV algebra and $a, b \in \mathcal{M}$. We say that a commutes with b ($a\mathbf{C}b$) iff $a \sqcap b = b \sqcap a$. The center of $\mathcal{M}$ is $\mathcal{C}(M) = \{a \in M | a\mathbf{C}b, \forall b \in \mathcal{M}\}$.

Similar to Lemma 3.2 and Lemma 3.3 in Giuntini (1998), we give the following results.

Lemma 4.2 *Let $\mathcal{M}$ be a wQMV algebra, then $a\mathbf{C}b$ iff $a = a \odot b^* \oplus (a \sqcap b)$.*

Proof Note that if $a\mathbf{C}b$, then $a = a \odot b^* \oplus (b \sqcap a) = a \odot b^* \oplus (a \sqcap b)$ by Proposition 2.7 (v). Conversely, assume $a = a \odot b^* \oplus (a \sqcap b) = a \odot b^* \oplus (b \sqcap a)$. Note that $a \odot b^* \preceq b^* \preceq (a \sqcap b)^*$ and $a \odot b^* \preceq a \odot b^* \oplus a^* = (b \sqcap a)^*$, thus $a \sqcap b = b \sqcap a$ by the cancelation law. □

Lemma 4.3 *Let $\mathcal{M}$ be a wQMV algebra, then $\mathcal{C}(M)$ is closed under *.*

Proof $\forall a \in \mathcal{C}(M)$ and $\forall b \in \mathcal{M}$, $a \sqcup b = a \odot b^* \oplus b = a \odot b^* \oplus b \odot a^* \oplus a \sqcap b = b \odot a^* \oplus a = b \sqcup a$. That is $a^*\mathbf{C}b^*$. Since $M = \{a^* | a \in M\}$, so $a^* \in \mathcal{C}(\mathcal{M})$. □

Lemma 4.4 *Let $\mathcal{M}$ be a wQMV algebra,*

(i) *$\forall a, b, c \in \mathcal{M}$, if $a\mathbf{C}b$ and $a\mathbf{C}c$, then $(a \sqcap b) \sqcap c = b \sqcap (a \sqcap c)$.*
(ii) *$\forall a, b, c \in \mathcal{M}$, if $b\mathbf{C}c$, then $(a \sqcap b) \sqcap c = a \sqcap (c \sqcap b)$.*
(iii) *$\forall a, b, c \in \mathcal{M}$, if $a\mathbf{C}b$, $a\mathbf{C}c$ and $b\mathbf{C}c$, then $(a \sqcap b) \sqcap c = c \sqcap (a \sqcap b)$.*

Proof (i). By Proposition 2.11, $(a \sqcap b) \sqcap c = (b \sqcap a) \sqcap c = (b \sqcap a) \sqcap (a \sqcap c) = (b \sqcap a) \sqcap (c \sqcap a) = b \sqcap (c \sqcap a) = b \sqcap (a \sqcap c)$.

(ii). By Proposition 2.11, $(a \sqcap b) \sqcap c = (a \sqcap b) \sqcap (b \sqcap c) = (a \sqcap b) \sqcap (c \sqcap b) = a \sqcap (c \sqcap b)$.

(iii). It follows (i) and (ii) that $(a \sqcap b) \sqcap c = a \sqcap (c \sqcap b) = a \sqcap (b \sqcap c) = (a \sqcap c) \sqcap b = c \sqcap (a \sqcap b)$.

□

Lemma 4.5 *Let $\mathcal{M}$ be a wQMV algebra. $\forall a, b \in \mathcal{C}(M)$ such that $a \preceq b^*$, $a \oplus b \in \mathcal{C}(M)$.*

Proof By Lemma 4.2, it only needs to show that $c = c \odot (a \oplus b)^* \oplus (c \sqcap (a \oplus b))$ $\forall c \in \mathcal{M}$. First, it is easy to conclude that (wQMV8) is equivalent to

$$a \oplus (b \sqcap c) = (a \oplus b) \sqcap (a \oplus c) \text{ if } a \preceq c^* \qquad \text{(WQMV8')}$$

And in fact $(b \oplus c)\mathbf{C}(b \oplus a)$:

$$\begin{aligned}
(b \oplus c) \sqcap (b \oplus a) &= b \oplus ((c \sqcap b^*) \sqcap (a \sqcap b^*)) && \text{(wQMV7)}\\
&= b \oplus ((c \sqcap b^*) \sqcap a)\\
&= b \oplus (a \sqcap (c \sqcap b^*))\\
&= b \oplus ((a \sqcap b^*) \sqcap (c \sqcap b^*))\\
&= (b \oplus a) \sqcap (b \oplus c) && \text{(wQMV7)}
\end{aligned}$$

Thus

$$\begin{aligned}
c &= c \odot a^* \oplus (c \sqcap a) && \text{(Lemma 4.2)}\\
&= (c \odot a^*) \odot b^* \oplus ((c \odot a^*) \sqcap b) \oplus (c \sqcap a) && \text{(Lemma 4.2)}\\
&= c \odot (a \oplus b)^* \oplus ((c \sqcap a) \oplus c \odot a^*) \sqcap ((c \sqcap a) \oplus b) && \text{(WQMV8')}\\
&= c \odot (a \oplus b)^* \oplus (c \sqcap ((c \sqcap a) \oplus b)) && \text{(Lemma 4.2)}\\
&= c \odot (a \oplus b)^* \oplus (c \sqcap ((b \oplus c) \sqcap (b \oplus a))) && \text{(WQMV8')}\\
&= c \odot (a \oplus b)^* \oplus (((b \oplus c) \sqcap c) \sqcap (b \oplus a)) && \text{(Lemma 4.4 (i))}\\
&= c \odot (a \oplus b)^* \oplus (c \sqcap (b \oplus a))
\end{aligned}$$

Therefore $(a \oplus b)\mathbf{C}c$. □

Theorem 4.6 *Let $\mathcal{M}$ be a wQMV algebra. The structure $\mathcal{C}(\mathcal{M}) = (\mathcal{C}(M), \oplus, *, \mathbf{1}, \mathbf{0})$ is an MV subalgebra of $\mathcal{M}$. $\forall a, b \in \mathcal{C}(\mathcal{M})$; the greatest lower bound (least upper bound) in $\mathcal{C}(\mathcal{M})$ is $a \sqcap b$ ($a \sqcup b$). $(\mathcal{C}(\mathcal{M}), \sqcap, \sqcup, *, \mathbf{1}, \mathbf{0})$ is a De Morgan distributive sublattice of the involutive bounded poset $(M, \preceq, *, \mathbf{1}, \mathbf{0})$.*

Proof By Lemmas 4.3 and 4.5, $\mathcal{C}(\mathcal{M})$ is closed under $\oplus$ and $*$. Furthermore, (wQMV7) coincides with (QMV7) in $\mathcal{C}(\mathcal{M})$; it follows that $\mathcal{C}(\mathcal{M})$ is a QMV subalgebra of $\mathcal{M}$.

Thus $\mathcal{C}(\mathcal{M})$ is an MV subalgebra of $\mathcal{M}$ by Theorem 3.2 of Giuntini (1998).

Assume $\forall a, b \in \mathcal{C}(\mathcal{M})$, $a \sqcap b \preceq a, b$. For any $c \in \mathcal{C}(\mathcal{M})$ such that $c \preceq a, b$, $c \preceq a \sqcap b$ by Proposition 2.9. So $a \sqcap b$ is the greatest lower bound of a and b in $\mathcal{C}(\mathcal{M})$. In the same way $a \sqcup b$ is the least upper bound of a, b in $\mathcal{C}(\mathcal{M})$. By Chang (1958) we also obtain that $(\mathcal{C}(\mathcal{M}), \sqcap, \sqcup, *, \mathbf{1}, \mathbf{0})$ is a De Morgan distributive sublattice of the involutive bounded poset $(M, \preceq, *, \mathbf{1}, \mathbf{0})$. □

5 Coupled bimonoids

In order to find a most general framework for developing axiomatic quantum mechanics, a study of ring-like structures was initiated. It was shown that semirings and MV algebras are closely related (Gerla 2003). In detail, a coupled semiring induces an MV algebra and conversely an MV algebra induces a pair of lc-semiring and dual lc-semiring which comprise a coupled semiring. The above coupled semiring is commutative (Gerla 2003). Later, the above result was generalized to semirings and pseudo-MV algebras which are non commutative algebraic structures (Shang and Lu 2007). The authors proved that right (left) coupled semiring contains right (left) pseudo-MV algebra and right (left) pseudo-MV algebra induces right (left) coupled semiring. In this section we demonstrate the similar relation between coupled bimonoids and l-wQMV algebras.

A strong bimonoid $\mathcal{K} = (M, \oplus, \mathbf{0}, \odot, \mathbf{1})$ is an algebraic structure where $\mathbf{0}$ and $\mathbf{1}$ are distinct elements of M; $\oplus$ and $\odot$ are binary operations on R satisfying:

(i) $(M, \oplus)$ is a commutative monoid with identity $\mathbf{0}$;
(ii) $(M, \odot)$ is a monoid with identity $\mathbf{1}$;
(iii) $\forall r \in M, \mathbf{0} \odot r = r \odot \mathbf{0} = \mathbf{0}$.

We call a strong bimonoid $\mathcal{K}$ right distributive if $(a \oplus b) \odot c = a \odot c \oplus a \odot b$ $\forall a, b, c \in M$. A strong bimonoid $\mathcal{K}$ is left distributive if $a \odot (b \oplus c) = a \odot b \oplus a \odot c$ $\forall a, b, c \in M$. Obviously, a strong bimonoid $\mathcal{K}$ is a semiring iff $\odot$ distributes over $\oplus$. We say that $\mathcal{K}$ is a commutative strong bimonoid if $(M, \odot)$ is a commutative monoid.

Example 5.1 As an example for strong bimonoid, consider $\mathcal{K} = (\mathbf{R}, \oplus, 0, \odot, 1)$. The operation $x \oplus y \overset{def}{=} x + y + xy$ and $x \odot y$ is taken as the usual product of real number.

Definition 5.2 A strong bimonoid $\mathcal{K} = (M, \oplus, \mathbf{0}, \odot, \mathbf{1})$ is called lattice-ordered (ls-bimonoid for short) iff it has the structure of a lattice such that for all $a, b \in \mathcal{K}$:

(i) $a \oplus b = a \vee b$;
(ii) $a \odot b \leq a \wedge b$.

Definition 5.3 A strong bimonoid $\mathcal{K} = (M, \oplus, \mathbf{0}, \odot, \mathbf{1})$ is called dual lattice-ordered (dual ls-bimonoid for short) iff it has the structure of a lattice such that for all $a, b \in \mathcal{K}$:

(i) $a \oplus b = a \wedge b$;
(ii) $a \odot b \geq a \vee b$.

ls-bimonoids and dual ls-bimonoids are additively idempotent.

Let R and S be strong bimonoids. A transposition morphism between R and S is a map $f : R \to S$ such that

(i) $f(\mathbf{0}_R) = \mathbf{0}_S$, $f(\mathbf{1}_R) = \mathbf{1}_S$.
(ii) $f(r \oplus r') = f(r) \oplus f(r')$ and $f(r \odot r') = f(r) \odot f(r')$ for any $r, r' \in R$.

If f is a transposition bimorphism, it is called a transposition isomorphism.

Definition 5.4 A coupled bimonoid $\mathcal{A}$ is a structure $(\mathcal{K}_1, \mathcal{K}_2, *)$ satisfying

(CB1) $\mathcal{K}_1 = (M, \vee, \mathbf{0}, \odot, \mathbf{1})$ and $\mathcal{K}_2 = (M, \wedge, \mathbf{1}, \oplus, \mathbf{0})$ are a commutative ls-bimonoid and a dual commutative ls-bimonoid, respectively,
(CB2) $* : M \longrightarrow M$ is a transposition isomorphism from $\mathcal{K}_1$ to $\mathcal{K}_2$; the image of a is denoted by a^*.
(CB3) $(a^*)^* = a$ for any $a \in M$.
(CB4) For all $a, b \in M$: $a \leq b$ iff $a = (a \oplus b^*) \odot b$.

Remark 5.5 $\forall a, b \in \mathcal{A}$, $a \leq b \Leftrightarrow a = a \wedge b \Leftrightarrow a^* = (a \wedge b)^* \Leftrightarrow a^* = a^* \vee b^* \Leftrightarrow b^* \leq a^*$.

Proposition 5.6 *Let $\mathcal{A} = (\mathcal{K}_1, \mathcal{K}_2, *)$ be a coupled bimonoid, where $\mathcal{K}_1 = (M, \vee, \mathbf{0}, \odot, \mathbf{1})$ and $\mathcal{K}_2 = (M, \wedge, \mathbf{1}, \oplus, \mathbf{0})$; then, $(M, \oplus, *, \mathbf{0}, \mathbf{1})$ is an S-algebra.*

Proof (S1), (S2), (S4) and (S6) are satisfied since $\mathcal{K}_2$ is commutative strong bimonoid. (S5) is satisfied by the definition of $*$.

Since $\mathbf{0} \leq x$ for any $x \in M$, by (CPS4) and (S4), $\mathbf{0} = (\mathbf{0} \oplus a^*) \odot a = a^* \odot a = (a \oplus a^*)^*$. Thus $a \oplus a^* = \mathbf{1}$, that is (S3). □

The following operations can be defined:

$$a \sqcap b = (a \oplus b^*) \odot b$$
$$a \sqcup b = (a \odot b^*) \oplus b$$

Obviously $(a \sqcap b)^* = a^* \sqcup b^*$. With these operations we can rewrite (CPS4) as "$a \leq b$ iff $a = a \sqcap b$." Note that $a \leq b \Leftrightarrow a \vee b = b \Leftrightarrow a^* \wedge b^* = b^* \Leftrightarrow b^* \leq a^*$. Thus (CPS4) again could be written as "$a \leq b$ iff $b = b \sqcup a$." If $a \leq b$, then $b \sqcap a = (b \sqcup a) \sqcap a = (a^* \odot b \boxplus a \boxplus a^*) \odot a = a$.

Proposition 5.7 *Let $\mathcal{A} = (\mathcal{K}_1, \mathcal{K}_2, *)$ be a coupled bimonoid. The following properties hold:*

(i) $a \oplus (b \sqcap a^*) = a \oplus b$
(ii) $a = (b^* \sqcap a) \oplus (a \odot b)$

Proof (i). $a \leq a \vee b \leq a \oplus b \Rightarrow a \oplus b = (a \oplus b) \sqcup a = [(a \oplus b) \odot a^*] \oplus a = (b \sqcap a^*) \oplus a$.
(ii). $a \odot b \leq a \wedge b \leq a \Rightarrow a = a \sqcup (a \odot b) = [a \odot (a \odot b)^*] \oplus (a \odot b) = [a \odot (a^* \oplus b^*)] \oplus (a \odot b) = (b^* \sqcap a) \oplus (a \odot b)$. □

The monotony of operations in coupled bimonoid could be verified in a similar way with Proposition 2.9; we omit the details here.

Proposition 5.8 (Monotony) *Let $\mathcal{A} = (\mathcal{K}_1, \mathcal{K}_2, *)$ be a coupled bimonoid, where $\mathcal{K}_1 = (M, \vee, \mathbf{0}, \odot, \mathbf{1})$ and $\mathcal{K}_2 = (M, \wedge, \mathbf{1}, \oplus, \mathbf{0})$. The monotonicity of several operations hold:*

(i) *If $a \leq b$, then $a \oplus c \leq b \oplus c$ for any $c \in M$.*
(ii) *If $a \leq b$, then $a \odot c \leq b \odot c$ for any $c \in M$.*
(iii) *If $a \leq b$, then $a \sqcap c \leq b \sqcap c$ for any $c \in M$.*
(iv) *If $a \leq b$, then $a \sqcup c \leq b \sqcup c$ for any $c \in M$.*

Theorem 5.9 *Let $\mathcal{A} = (\mathcal{K}_1, \mathcal{K}_2, *)$ be a coupled bimonoid, where $\mathcal{K}_1 = (M, \vee, \mathbf{0}, \odot, \mathbf{1})$ and $\mathcal{K}_2 = (M, \wedge, \mathbf{1}, \oplus, \mathbf{0})$. The structure $\mathcal{M} = (M, \oplus, *, \mathbf{0}, \mathbf{1})$ is a l-wQMV algebra.*

Proof First $\mathcal{M}$ is an S-algebra. Note that $a \leq (a \oplus b) \sqcap (a \oplus c)$ by Proposition 5.8 (iii), and from Proposition 5.7 we obtain

$$(a \oplus b) \sqcap (a \oplus c) = a \oplus \left[(b \sqcap a^*) \sqcap (c \sqcap a^*)\right] \quad \text{(wQMV7)}$$

along with the same the reasoning of Theorem 2.10. Furthermore, $\mathcal{M}$ is a l-wQMV algebra for the partial order $\leq$ coincides with $\preceq$ by (CPS4). □

Theorem 5.10 *Let $\mathcal{M} = (M, \oplus, *, \mathbf{0}, \mathbf{1})$ be a l-wQMV algebra. The reducts $\mathcal{K}_1 = (M, \vee, \mathbf{0}, \odot, \mathbf{1})$ and $\mathcal{K}_2 = (M, \wedge, \mathbf{1}, \oplus, \mathbf{0})$ are a commutative ls-bimonoid and a dual commutative ls-bimonoid, respectively, and $(\mathcal{K}_1, \mathcal{K}_2, *)$ is a coupled bimonoid.*

Proof

(CB1) : In $\mathcal{M}$, the reducts $(M, \vee, \mathbf{0})$, $(M, \odot, \mathbf{1})$, $(M, \wedge, \mathbf{1})$ and $(M, \oplus, \mathbf{0})$ are commutative monoids. Further, $a \odot \mathbf{0} = \mathbf{0} \odot a = \mathbf{0}$ and $a \oplus \mathbf{1} = \mathbf{1} \oplus a = \mathbf{1}$ for any $a \in M$. So $(M, \vee, \mathbf{0}, \odot, \mathbf{1})$ is a ls-bimonoid, and $(M, \wedge, \mathbf{1}, \oplus, \mathbf{0})$ is the dual ls-bimonoid.
(CB2), (CB3) : By the definitions of $*$ and $\odot$ in $\mathcal{M}$.
(CB4) : By the definition of $\preceq$ in $\mathcal{M}$. □

Let $\mathcal{A} = (\mathcal{K}_1, \mathcal{K}_2, *)$ be a coupled bimonoid, $\mathcal{K}_1 = (M, \vee, \mathbf{0}, \odot, \mathbf{1})$ and $\mathcal{K}_2 = (M, \wedge, \mathbf{1}, \oplus, \mathbf{0})$. Recall that a residuated lattice is a structure $\mathcal{L} = (L, \leq, \odot, \mathbf{1})$ such that

(i) $(L, \leq)$ is a lattice.
(ii) $(L, \odot, \mathbf{1})$ is a monoid.
(iii) Residuation properties: For all c, there exists a greatest b for every a and exists a greatest a for every b, such that $a \odot b \leq c$.

Alternatively the residuation properties could be stated as there exists a binary operation $\rightarrow$ being the right adjoint to the operation $\odot$:

$$a \odot b \leq c \Leftrightarrow b \leq a \rightarrow c \tag{1}$$

$(M, \leq, \odot, \mathbf{1})$ is a residuated lattice if Eq. (1) is satisfied in $\mathcal{K}_1$. So the distributivity law

$$a \odot (b \vee c) = (a \odot b) \vee (a \odot c) \tag{2}$$

is true as a consequence of the residuated lattice. Since $\mathcal{K}_1$ is commutative, it becomes a lattice-ordered commutative semiring if endowed with Eq. (1) the corresponding l-wQMV algebra turns out to be an MV algebra. Conversely, $\mathcal{K}_1$ is a lattice-ordered commutative semiring if Eq. (2) holds in $\mathcal{K}_1$, then $\mathcal{A}$ corresponds to an MV algebra, so Eq. (1) is satisfied. That is, Eqs. (1) and (2) are equivalent in coupled bimonoid $\mathcal{A}$. Therefore coupled bimonoids are exactly the generalization of coupled semirings which do not satisfy Eq. (2), as well as l-wQMV algebras are exactly the generalization of MV algebras which do not satisfy Eq. (1).

For a QMV algebra, it is difficult to give its corresponding ring-like structure directly. In the following, we can deduce the ring-like structures corresponding to l-QMV algebras. By Theorems 2.14, 5.9 and 5.10, we define the following algebras:

Definition 5.11 A strong coupled bimonoid $\mathcal{A}$ is a structure $(\mathcal{K}_1, \mathcal{K}_2, *)$ satisfying

(SCB1) $\mathcal{K}_1 = (M, \vee, \mathbf{0}, \odot, \mathbf{1})$ and $\mathcal{K}_2 = (M, \wedge, \mathbf{1}, \oplus, \mathbf{0})$ are a commutative ls-bimonoid and a dual commutative ls-bimonoid, respectively,
(SCB2) $* : M \longrightarrow M$ is a transposition isomorphism from $\mathcal{K}_1$ to $\mathcal{K}_2$; the image of a is denoted by a^*.
(SCB3) $(a^*)^* = a$ for any $a \in M$.
(SCB4) For all $a, b \in M$: $a \leq b$ iff $a = (a \oplus b^*) \odot b$.
(SCB5) For all $a, b \in M$: $a \oplus b = a \oplus (a \odot b)^* \odot b$.

Corollary 5.12 *Let* $\mathcal{A} = (\mathcal{K}_1, \mathcal{K}_2, *)$ *be a strong coupled bimonoid, where* $\mathcal{K}_1 = (M, \vee, \mathbf{0}, \odot, \mathbf{1})$ *and* $\mathcal{K}_2 = (M, \wedge, \mathbf{1}, \oplus, \mathbf{0})$. *The structure* $\mathcal{M} = (M, \oplus, *, \mathbf{0}, \mathbf{1})$ *is a l-QMV algebra.*

Corollary 5.13 *Let* $\mathcal{M} = (M, \oplus, *, \mathbf{0}, \mathbf{1})$ *be a l-QMV algebra. The reducts* $\mathcal{K}_1 = (M, \vee, \mathbf{0}, \odot, \mathbf{1})$ *and* $\mathcal{K}_2 = (M, \wedge, \mathbf{1}, \oplus, \mathbf{0})$ *are a commutative ls-bimonoid and a dual commutative ls-bimonoid, respectively. And* $(\mathcal{K}_1, \mathcal{K}_2, *)$ *is a strong coupled bimonoid.*

6 Conclusion

In this paper, we propose a new quantum structure—wQMV algebras—and find that it can share most of the important properties with QMV algebras except for some idempotent properties. Furthermore, we give the concept of ring-like structures coupled bimonoids and establish the corresponding relation between l-wQMV algebras and the coupled bimonoids, which is similar to the relation between MV algebras and coupled semirings. Interestingly, since coupled bimonoids could be obtained by removing distributivity from coupled semirings, correspondingly, it is easy to see that l-wQMV algebras could also be viewed as a structure by removing distributivity from MV algebras. For a lattice-ordered QMV algebras, it is very difficult to construct its corresponding ring-like structure directly. By the use of the wQMV algebras and coupled bimonoids, we obtain the ring-like algebraic structure corresponding to a l-QMV algebra. Certainly, with the similar means in Shang et al. (2009, 2012), we could also set up computation theory based on l-wQMV algebras.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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